Filling a Reservoir

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March 12, 2019

Abstract

This paper aims to establish relationships between the outflow and the inflow of water in a general reservoir in the United Kingdom, through the use of ordinary differential equations.

Two proposed models of the inflow-outflow equations include assuming the reservoir as a perfect cuboid, and that it is at 50% of its maximum capacity. The Derwent reservoir in Derbyshire is considered to be an ideal reservoir to model first in this paper and calculations for the daily inflow and outflow give us a rate of change equal to $2.38 \times 10^{-2}\ m\ day^{-1}$.

Our second model looks more closely into these same factors, taking into account further detail where possible. The Manning-Strickler equation is used to model the inflow of water as a good approximation for open water channels.

Assuming a laminar flow, using Hamill's equation, a function of the height of the reservoir is a good enough fit for the outflow through pipes. Shuttleworth's modified equation takes into account the temperature in the region to give an approximation for the outflow by evaporation. Taking into account the rainfall between 1980 and 2010, the average precipitation found was 1120 mm equivalent to a flow of 3.55×10^{-8} ms^{-1}.

1 Introduction

Reservoirs and artificial lakes have been constructed since 5th century BC [Wilson and Wilson, 2005] and, as technology and our understanding of fluids and forces has evolved, their complexity and efficiency have grown and developed similarly.

The idea of collecting rainwater and water from linking rivers and storing it in surplus has proven essential in areas such as agriculture and farming throughout history.

In the modern era, however, almost all reservoirs play a fundamental role in generating electricity by taking advantage of hydroelectric power.

In any case, it is vital to know the volume of water that is in the reservoir at any given time, how much the water varies in time and also whether it is increasing or decreasing. If this is not taken into account then a reservoir or dam runs the risk of either draining or overflowing - both of which have serious financial and economic detriment to the surrounding community.

Given the global scale of reservoirs and their consistent presence throughout history, there are currently existing sources on this topic, both scientific reports and online articles. By choosing the appropriate data from these, we look at creating a functional differential equation to model the rate of change of water in a reservoir with respect to time.

2 Methods and Results

When modelling the rate of change of water in a reservoir as a function of time, we can break down the problem into how much water enters the system (inflow) and how much leaves the system (outflow). These in turn both contain two factors.

For inflow the primary factors are rainfall and linking rivers while for outflow the rate of evaporation and how much water is provided for use in the community are the primary factors.

First Model

Assumptions

- The reservoir is constructed in the shape of a cuboid - when dealing with volumes, a constant area results in a linear change in height.
- The reservoir is at 50% capacity - the data we have used for evaporation relies on this.
- Both inflow and outflow occur at a constant rate - this will avoid any exponential equation and gives us a general idea if the reservoir increases or decreases with time.
- Derwent reservoir is representative of all reservoirs - all reservoirs vary in size and data for inflow and outflow is particularly prominent for this particular reservoir.

Method

With the length, area and volume of the reservoir[Windows2universe.org.,2019] known, we calculate the reservoir's remaining measurements, where h is the height of the water.

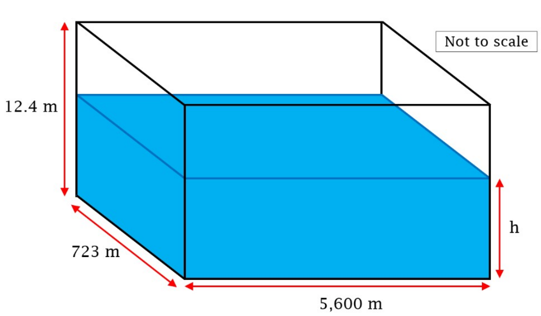

Figure 1: *Diagram showing the structure and measurements of our model reservoir.*

We can now model the rate of change of the height of the water with the following differential equation:

$$\frac{dh}{dt} = U - K$$

where;

U = Inflow
K = Outflow

80% of rainfall is either directly over the ocean or flows into the ocean after falling on land first [Nwl.co.uk, 2019]. By applying this principle to our reservoir and its catchment area we calculate the total inflow of water into the reservoir using the following calculations:

1. Taking the total area of the reservoir and the catchment area to be 28,000 acres [Nwl.co.uk, 2019]and the total rainfall in a year, specific to this catchment area, to be 953 mm [Nwl.co.uk, 2019] we calculate the total volume of water that falls to be $1.08 \times 10^8\ m^3$.
2. By taking 80% of this value, we calculate that in one year, $8.64 \times 10^7\ m^3\ year^{-1}$ of water enters the reservoir.
3. Dividing this by 365 gives us $2.37 \times 10^5\ m^3\ day^{-1}$.
4. Dividing the daily increase in volume by the area of the reservoir ($4.047 \times 10^6\ m^2$) gives us an increase in height of $5.85 \times 10^{-2}\ m\ day^{-1}$.

This value includes both direct rainfall and water from rivers and surface runoff and so is our total inflow (U) to the reservoir each day.

The volume of water that is pumped out of the reservoir per day is $1.386 \times 10^6\ m^3$[Nwl.co.uk, 2019].

We simply divide this by the area of the reservoir($4.047 \times 10^6\ m^2$) to calculate the decrease in height per day. This gives us a value of $3.42 \times 10^{-2}\ m$.

The total amount of water that evaporates from all artificial lakes and reservoirs around the world is 346 km^3 [Kohli and Frenken, 2015]. Using this statistic, we calculate the outflow of water due to evaporation using the following steps:

1. Calculate the volume of water that evaporates from each reservoir in a year by dividing our original value by the total number of artificial lakes and reservoirs in the world, 515,176 [John et al., 2006].
2. Divide this value by 365 to give us a daily rate of 1840 m^3.
3. Divide this value by the area of our reservoir($2.047 \times 10^6\ m^2$) to give us a decrease in height of $4.52 \times 10^{-4}\ m$.

Substituting our value for inflow into U and the summation of our two values for outflow into K, we have our differential equation:

$$\frac{dh}{dt} = 5.85 \times 10^{-2} - (3.42 \times 10^{-2} + 4.52 \times 10^{-4})$$

$$\frac{dh}{dt} = 2.38 \times 10^{-2}$$

This then integrates analytically to:

$$h(t) = 2.38 \times 10^{-2} \cdot t + C$$

By taking the initial case that at time t = 0 $days$, h = 6.2 m, we have the following equation of h with respect to time:

$$h(t) = 2.38 \times 10^{-2} \cdot t + 6.2$$

Evaluation of First Model

We can see by our differential equation that, if left long enough, our reservoir will eventually reach maximum capacity and then begin to overflow. To find out when this will occur we equate h(t) to the height of our reservoir (12.4) and solve for t. This gives us a value of 260.5 days.

To further analyse these results we plot the function h(t) $0 < t < 260.5$ into GeoGebra which can be seen in figure 2.

In reality, any reservoir or dam should neither overflow nor drain. The fact that our solution suggests that our model reservoir would overflow as time increases means it is not entirely accurate.

This model was done to give us an idea of what would happen to the height of the reservoir if the inflow and outflow were set at a constant rate and the reservoir was then left unattended.

The fact that our differential equation is positive, however, indicates that the reservoir gains more water than it loses and uses, which is ideal when constructing a reservoir or dam.

Second Model

For our second model we attempt to create a more defined and accurate solution.

We do this by focusing on the same factors for inflow and outflow as before but instead further break them down in order to take into account additional factors.

Assumptions

- There is only one single main river supplying water to the reservoir. Any small streams and water from surface run off is negligible.
- Fluid flow through pipes is laminar (Reynolds number less than or equal to 2300) - turbulent flows are harder to predict and do not occur all of a sudden from laminar flows, more often there is region where the flow fluctuates before becoming turbulent [Litvinov, 2011] A laminar flow can be visualised as having smooth streamlines and having an ordered motion of flow.
- Pipes responsible for removing water are placed level with the bed of the reservoir - simplifies the height of water from the base of the pipes to the surface of the reservoir to be equal to h.

Method

The inflow of water due to rivers is calculated on the basis of the Manning-Strickler equation [Manning, 1891] [Manning, 1895] [Strickler, 1923] which relates the flow rate of a fluid in open channels to the hydraulic radius of the channel and its slope. The equation, chosen for its simplicity in determining flow rates, can be shown:

$$V = \frac{1}{n} \cdot R^{\frac{2}{3}} \cdot S^{\frac{1}{2}}$$

where;

V = Flow rate (ms^{-1})
R = Hydraulic radius
S = Channel slope
n = Manning's coefficient (Chow, 1959)

An advantage of the Manning-Strickler equation is the fact that n is independent to the depth of flow for fully turbulent flow over a rough surface [Yen, 2002] and can be used to generalise the velocity for any river in the UK.

The average precipitation in the UK from the years 1980-2010 has been 1120 mm [Barker et al., 2019]. This is fairly accurate as there has been an insignificant amount of change in the mean and extreme precipitation values recorded for England and Wales from 1766 to 2011 wherein lies the 30 year period stated above.

There is an upward trend in precipitation values in winter, however, this is countered by a dip in same values during summer [Simpson and Jones, 2013] and therefore their effects cancel out.

Converting the annual precipitation into SI units, the 1120 mm precipitation is equivalent to 3.5515×10^{-8} ms^{-1}.

The outflow is considered to be similar to the flow of water through pipes in this model. A laminar flow can be visualised as having smooth streamlines and an ordered motion of flow.

Considering the outflow to be laminar and a large and sharp lateral aperture, the velocity of flow can be modelled as:

$$V = \frac{2}{3} \cdot \frac{C_d \cdot (2 \cdot g)^{\frac{1}{2}} \cdot (H_2^{\frac{3}{2}} - H_1^{\frac{3}{2}})}{H_2 - H_1}$$

where;

V = Flow rate (ms^{-1})
C_d = Discharge coefficient
g = Acceleration due to gravity
H_1 = Length from top of orifice to reservoir surface
H_2 = Length from bottom of orifice to reservoir surface

C_d can be taken as 0.62 when the exact value is not known but only for sharp orifices [Daugherty and Franzini, 2006] and g is known to be 9.81 ms^{-1}.H_2 is simply equal to h and to calculate H_1 we take 2 pipes, both of diameter 1.04 m [Nwl.co.uk, 2019] and turn this into one big single pipe to simplify the model. This gives us a diameter of 1.47 m and so H_1 = h - 1.47.

By substituting these values into the equation:

$$V = \frac{1.83 \cdot (h^{\frac{3}{2}} - (h - 1.47)^{\frac{3}{2}})}{h - (h - 1.47)}$$

Thus, the equation simplifies to:

$$V = 1.24 \cdot (h^{\frac{3}{2}} - (h - 1.47)^{\frac{3}{2}})$$

The Penman equation [Penman, 1948] describes the rate of evaporation from open water bodies.

A much simplified version of this equation developed by Shuttleworth [Shuttleworth, 2007] is given by

$$E_{mass} = \frac{(mR_n + 6.43\gamma(1 + 0.536V_{wind})\delta e)}{\lambda v(m + \gamma)}$$

where;

E_{mass} = the rate of evaporation (ms^{-1})
m = slope of the saturation vapour pressure curve given by

$$0.04145e^{0.06088T}$$

where T is the temperature in Celsius [ASAE Standards, 1998] [Priestley and Taylor, 1972]
R_n = Net irradiance which is 8.74368 MJ $m^{-2}day^{-1}$
This is based off 30 years of observed historical monthly average sunshine duration data [Burnett et al., 2014].
γ = Psychrometric constant given by

$$\frac{1.63 \times 10^{-3} P_{kPa}}{2.256}$$

[Allen et al., 1998]
V_{wind} = Wind speed can be considered to be in the range 4.3 to 5.3 ms^{-1} which was calculated over 30 years of data (1980-2010) [Earl et al., 2013]
δe = the vapour pressure deficit can be calculated by

$$\frac{100 - RH}{100} SVP$$

where RH is humidity percentage and SVP is the saturated vapour pressure [Murray, 1967]
λv = The latent heat of vaporization which is known to be 2.256 $MJkg^{-1}$ [Legates, 2005].

Assuming that the mean air pressure lies between the highest and lowest values ever recorded in the UK, the mean air pressure can be taken to be 989.6 hPa i.e. 98.96 kPa. Thus the value for the psychrometric constant (γ) can be taken to be 0.0715.

Thus the simplified equation is now:

$$E_{mass} = \frac{8.74m + 0.460(1 + 0.536V_{wind})\delta e}{2.256(m + 0.0715) \times 8.64 \times 10^7}$$

A table of values by Monteith and Unsworth [Monteith and Unsworth, 1990] show the correlation between temperature and the saturated vapour pressure.

Using the same idea as for the first model, the differential equation can be given by:

$$\frac{dh}{dt} = \frac{1}{n} \cdot R^{\frac{2}{3}} \cdot S^{\frac{1}{2}} - 1.24 \cdot (h^{\frac{3}{2}} - (h - 1.47)^{\frac{3}{2}}) - \frac{8.74m + 0.460(1 + 0.536V_{wind})\delta e}{2.256(m + 0.0715) \times 8.64 \times 10^7} + 3.55 \times 10^{-8}$$

Evaluation of Second method

This differential equation, though being much more accurate in the parameters chosen and evaluated separately, cannot be solved analytically. The use of this equation, however, can be to verify the methodology of proceeding with such a problem.

The given equation could be used as a guideline to understand the nuances of fluid flow problems in general. The equation lacks certain additional elements, the key being that it is quite limited to solving only the rate of outflow in smooth pipes. The frictional forces cannot be considered due to difficulty in simulating and solving the problem.

3 Discussion and Conclusion

Our second model is evidence that this problem can become highly elaborate when even only a few assumptions are removed and therefore demonstrates the difficult nature of this problem.

In our first model, while it provides a general solution that we know isn't entirely accurate, we can see that it is leading us in the right direction. The fact that the gradient of our equation for $h(t)$ is positive shows this and is what we would expect, given the assumptions we have made.

When constructing a reservoir, one would do so in an area in which you would expect to receive a greater amount of rainfall than the average for the region it is constructed in. By increasing the amount of inflow, you reduce the risk that a reservoir will decrease in height over time. While a reservoir running the risk of overflowing isn't necessarily practical either, it is more desired than its counterpart and draining completely and this is what our first model proves.

Furthermore, South West Water's current water storage statistics support this[SouthWestWater.co.uk, 2019].

This shows that their 5 reservoirs currently average at 88.6% capacity which is what our reservoir will increase to when starting from 50% capacity.

In reality, ideal reservoirs and dams are kept between a minimum and maximum height to avoid both overflowing or draining. The fundamental assumption we have made in our models is that both inflow and outflow are occurring at the same time and at a constant rate while in reality this is not the case. Should the rate of inflow decrease during a particular period then the amount of water pumped out would need to be reduced for the same period to accommodate this and vice versa.

This behaviour is extremely difficult to model but if we were to further improve our work we would need to begin to take into account the fact that inflow and outflow, in fact, occur at different times and are not constant. The function we should then expect to calculate would be of sinusoidal nature meaning that, while the height will always be changing, there isn't any risk of it ever exceeding a minimum and maximum value, specific to each individual reservoir.

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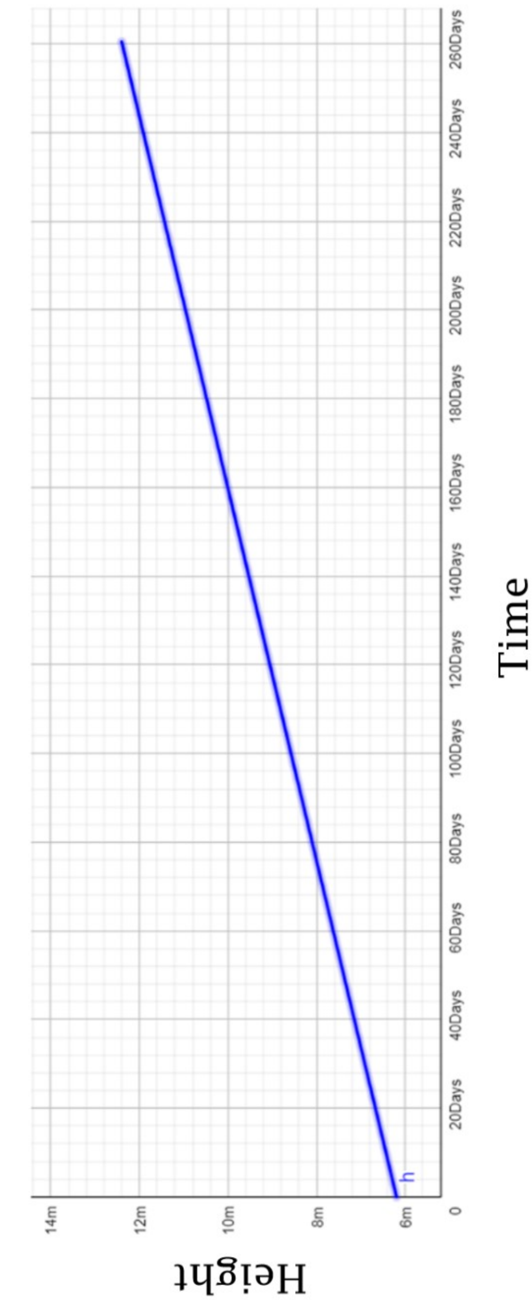

Figure 2: *Graph of h(t) where t is measured in days and height in meters assuming that the reservoir begins from 50% capacity and does not exceed 100% capacity.*