University of BRISTOL

Voting Models in Random Networks

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Introduction & motivation

- Networks of interacting agents are a key tool to model social phenomena.
- Fundamental applications to business and marketing decision making processes.
- Pioneered by the work by Watts and Strogatz [1] which shows common features in neural networks in worms, power grid networks and film actor networks.
- Uses in analysing the diffusion of innovations in social networks.
- Can local interaction between neighboring agents convince the population to conform to single opinions?
- How fast is the process of convincing the entire population?
- Depends on the method of agent interactions.

[1] Watts, Duncan J., and Steven H. Strogatz. "Collective dynamics of 'small-world'networks." *Nature* 393.6684 (1998): 440-442.

Proposed algorithm for opinion convergence

Voter algorithm

- Node in the network adopts the same behaviour as one of its randomly chosen neighbours.
- Original algorithm proposed in [2] but assumed that there are only two possible behaviours to adopt.
- Original contribution by authors: Do not make this assumption and allow all nodes to have different opinions initially.

Author assumptions:

- Finite number of nodes in the network
- The graph under consideration is strongly connected i.e. there is always a path between any 2 nodes.

[2] Liggett, Thomas M. “The Voter Model”. *Interacting Particle Systems*. Springer, Berlin, Heidelberg, (2005). 264-314.

General voter model setup

- Each node schedules the update of its own opinion through a Poisson process where $\lambda = 1$.
- Evolution of system tracked using discrete index k.
- Initially all nodes have different opinions, so each node assigned a different colour. $S = \{1, 2, 3 \ \ldots N\}$ where N is the number of nodes.
- The state of a node is indicated by $c_i[k]$ where i is the node index, at iteration k with unique colour c.
- The original voter model is a special case of the current model as $S = \{1, 2\}$.

Graph theory terminology

- Consider an undirected simple graph $G = (\upsilon, \varepsilon)\ \upsilon, set\ of\ nodes;\ \varepsilon, set\ of\ edges$
- Degree of node i (d_i), is the cardinality of the set of neighbours, N_i of each node i $\in V$
- Degree matrix $\boldsymbol{D}$ is where $D_{i,i} = d_i$.
- Adjacency matrix $\boldsymbol{A}$ where $A_{i,j} = A_{j,i} = 1$ if link exists between nodes $i,$ j.

Reaching agreement

- Initially at iteration $k = 0$, all nodes have different opinions.
- Number of neighbours of node i coloured with c at iteration k is given by $Q_{i,c}[k]$
- The updating rule:

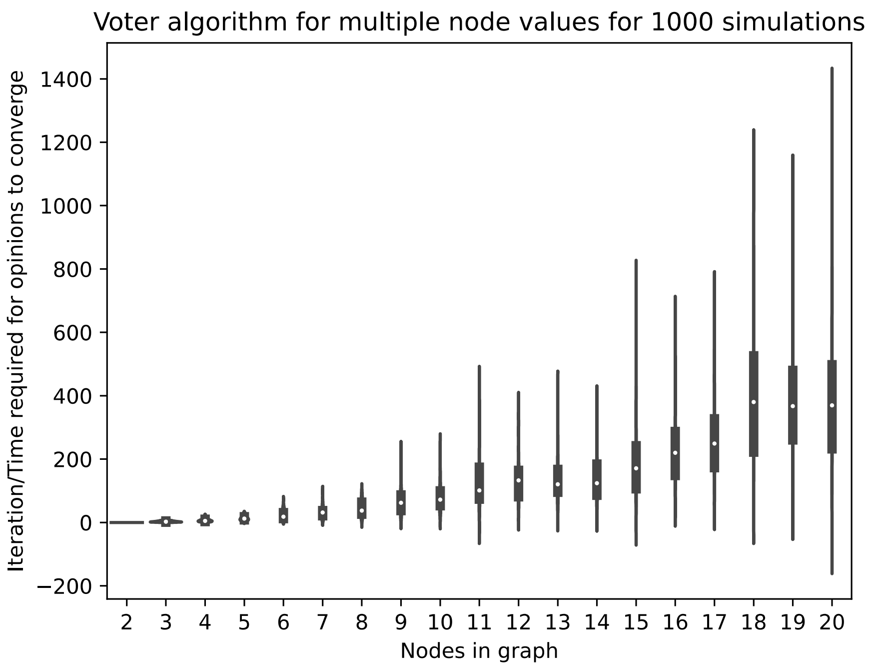
$$c_i[k+1] = X, \qquad X \sim \frac{1}{d_i} \sum_{c \in \{1,2\ldots N\}} Q_{i,c}[k]\delta(x-c)$$

- At each iteration, node i copies the opinion of one of its neighbours uniformly at random.

$$c_i[k+1] = X, \qquad X \sim \frac{1}{d_i} \sum_{j \in N_i} \delta(x - c_j[k])$$

Convergence properties

- Evolution of the system in time depends on a random update process.
- Depends on the most recent update.
- Viewed as a Markov process, with the absorbing state achieved when opinions converge to a single opinion.
- Proof of concept involves proving a single sequence of N consecutive updates leading to an absorbing state always exists.



Simulating the voter algorithm on a random graph

- Create a random strongly connected graph $G = (\upsilon, \varepsilon)$ where $\upsilon = 15, \varepsilon = 30$ with all nodes initially coloured differently.
- This graph required 134 updates for convergence.

Coalescing random walks

- Initially at $t = 0$, there exists a hypothetical particle at each vertex on G.
- Each particle jumps onto a neighbour independently according to a Poisson process with rate 1.
- When two or more particles meet at a node, they coalesce and form a single particle. This is a representation of multiple opinions converging to one opinion.
- These particles form a single random walk on G.
- Convergence of time of the voter model to a consensus and the convergence time of a coalescing random walk to a single particle has the same distribution.

Upper bound of convergence time for a coalescing random walk on G

- Let expected upper bound for convergence for a coalescing random walk be $E[T_{crw}]$
- Let mean hitting time for a random walk initialised at i and ending at j be $E_i[T_j]$.

Lemma 1

$$E[T_{crw}] \leq e\ln(N+2)\max_{i,j} E_i[T_j]$$

Lemma 2

$$E_i[T_j] = 2|\varepsilon| \sum_{k=2}^{N} \frac{1}{1-\lambda_k\left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{D}^{-\frac{1}{2}}\right)} \left(\frac{v_{kj}^2}{D_{jj}} - \frac{v_{kj}v_{ki}}{\sqrt{D_{jj}D_{ii}}}\right)$$

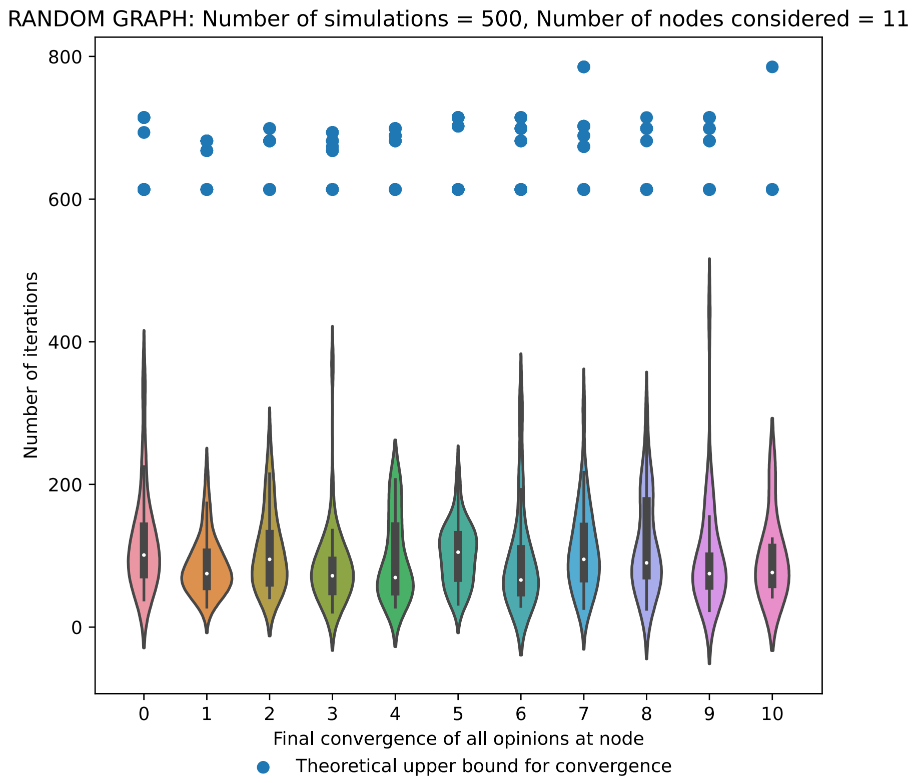
Where v_k and λ_k are the eigenvector and eigenvalue of $\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{D}^{-\frac{1}{2}}$

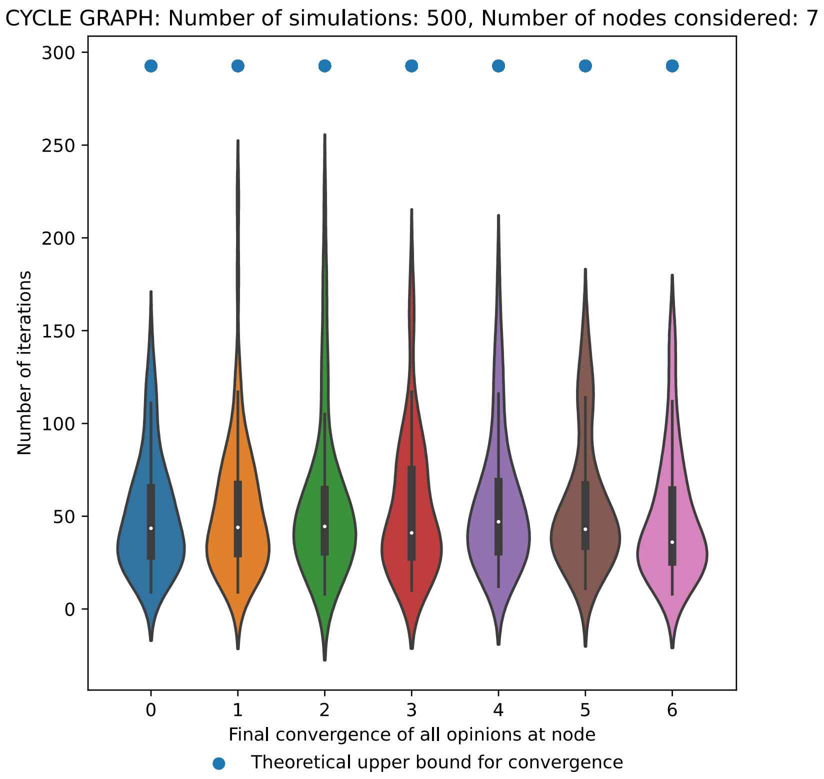
Main result

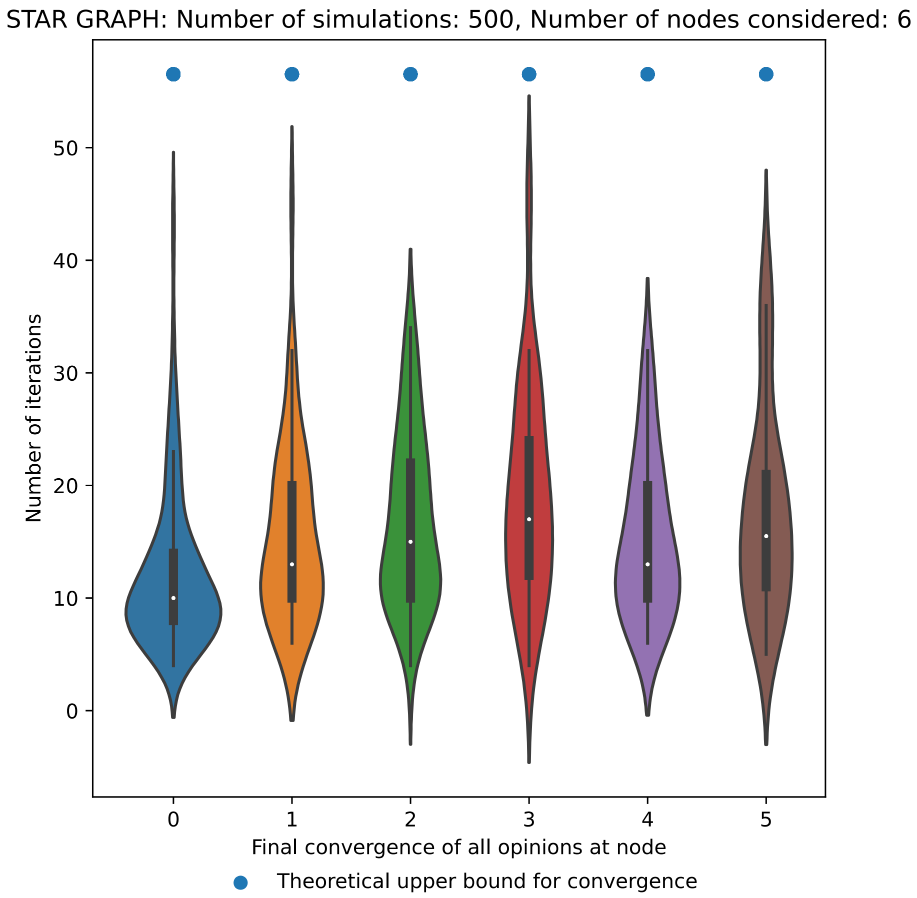
- Combining Lemmas 1 and 2, let $E[T_{crw}] = E[T]$ (duality):
- The symmetric reduced adjacency matrix $(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{D}^{-\frac{1}{2}})$ contains information about the probability that a random walk starting at node i will be at node j after t timesteps. Largest eigenvalue $\lambda_1 = 1$.

$$E[T] \leq \frac{4e \ln(N+2)|\varepsilon|}{1 - \lambda_2\left(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{A}\boldsymbol{D}^{-\frac{1}{2}}\right)} \max_j D_{jj}^{-1}$$

- Cycle graph: $E[T] \leq eN^2 \ln(N+2)$
- Star graph: $E[T] \leq e(2N-2) \ln(N+2)$


RANDOM GRAPH: Number of simulations = 500, Number of nodes considered = 11
Number of iterations
800
600
400
200
0
0 1 2 3 4 5 6 7 8 9 10
Final convergence of all opinions at node
Theoretical upper bound for convergence


CYCLE GRAPH: Number of simulations: 500, Number of nodes considered: 7
Number of iterations
300
250
200
150
100
50
0
0 1 2 3 4 5 6
Final convergence of all opinions at node
Theoretical upper bound for convergence



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Disagreement on a network

- For the model observed earlier, the update rule always leads to consensus of opinions.
- What would happen if this update rule was changed?
- Instead of node i waking up and choosing a neighbour at random, suppose the node chooses the colour/opinion with the majority in the neighbourhood.
- Called the Label Propagation Algorithm.

$$c_i[k+1] = \arg\max_c Q_{i,c}[k]$$

https://github.com/vedang-joshi/ComplexNetworks